SURJECTIVITY OF A GLUING FOR SPECIAL LAGRANGIAN SUBMANIFOLDS OF DIMENSION THREE WITH ISOLATED SINGULARITIES MODELLED ON THE CLIFFORD TORUS CONE

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Abstract: Let M be a compact special Lagrangian submanifold of dimension 3 with an isolated singularity at which M is tangent to a cone $C \subset \mathbb{R}^6$ with multiplicity 1. Suppose that $C \cap S^5$ is the Clifford torus cone. One can apply the gluing technique to M by a theorem of Joyce. One obtains then a compact non-singular special Lagrangian submanifold sufficiently close to M as varifolds in Geometric Measure Theory. The main result of this paper is as follows: all special Lagrangian varifolds sufficiently close to M are obtained by the gluing technique.

1. Introduction

Special Lagrangian submanifolds are area minimizing Lagrangian submanifolds discovered by Harvey and Lawson [8]. There is no obstruction to the C^1 deformation of compact special Lagrangian submanifolds by a theorem of Mclean [14, Theorem 3.6]. The moduli space of compact special Lagrangian submanifolds is therefore a manifold of finite dimension. One can compactify it using varifolds in Geometric Measure Theory. It is however difficult to understand general special Lagrangian varifolds. Joyce [13] has studied isolated singularities with multiplicity 1 smooth tangent cones.

Let $C \subset \mathbb{C}^3$ be the Clifford torus cone which will be defined in (4) below. One important property is that C is stable in the sense of Joyce [11, p11].

Suppose now that there exists a compact special Lagrangian submanifold M of dimension 3 with one point singularity where M is tangent to C with multiplicity 1. One can apply the gluing technique to M by a theorem of Joyce [13, Theorem 10.4]. Let L be a special Lagrangian submanifold of $\mathbb{C}^3$ tangent to C with multiplicity 1 at infinity in $\mathbb{C}^3$. One glues L to M in Joyce's theorem. One obtains then a compact non-singular special Lagrangian submanifold sufficiently close to M as varifolds in Geometric Measure Theory. The main result of this paper is as follows:

All special Lagrangian varifolds sufficiently close to M are obtained by the gluing technique; we shall give a more precise statement in what follows (see Theorem 6).

We give here a brief outline of the proof; see the latter part of this section for the details. The proof is similar to that of a theorem of Donaldson in the Yang–Mills theory; see Theorem 7 below. We first prove an analogue of Uhlenbeck's removable singularities theorem in the Yang–Mills theory. We use here an idea of a theorem of Simon [15, Corollary, p564], which proves the uniqueness of multiplicity 1 smooth tangent cones of minimal surfaces. We prove next the uniqueness of local models for desingularizing M using symmetry of C where M, C are as above. These are the main parts of the proof.

We shall now define C; see (4) below. Define $f : \mathbb{C}^3 \to \mathbb{R}^3$ by

$$f(z_1, z_2, z_3) = (|z_1|^2 - |z_3|^2, |z_2|^2 - |z_3|^2, \operatorname{Im} z_1 z_2 z_3) \tag{1}$$

where Im is defined as follows: if $z = x + \sqrt{-1}y$ with $x, y \in \mathbb{R}$ then $\operatorname{Im} z = y$. Put

$$\Omega_{\mathbb{C}^3} = dz^1 \wedge dz^2 \wedge dz^3. \tag{2}$$

One can then define $\Omega_{\mathbb{C}^3}$-special Lagrangian submanifolds of $\mathbb{C}^3$ as in Harvey and Lawson [8].

Theorem 1 (Harvey and Lawson [8, III.3.A, Theorem 3.1])**.** *Let $a \in \mathbb{R}^3$. Then $f^{-1}(a)$ is an $\Omega_{\mathbb{C}^3}$-special Lagrangian submanifold of $\mathbb{C}^3$ possibly with singularities; $f^{-1}(a)$ is singular if and only if $a \in Y \times \{0\}$ where*

$$Y = \{(x, x) : x \geq 0\} \cup \{(-x, 0) : x \geq 0\} \cup \{(0, -x) : x \geq 0\}; \tag{3}$$

notice here that $Y \subset \mathbb{R}^2$ and $Y \times \{0\} \subset \mathbb{R}^3$.

Put

$$C = f^{-1}(0, 0, 0) \cap \{(z_1, z_2, z_3) : \operatorname{Re} z_1 z_2 z_3 \geq 0\} \subset \mathbb{C}^3 \tag{4}$$

where Re is defined as follows: if $z = x + \sqrt{-1}y$ with $x, y \in \mathbb{R}$ then $\operatorname{Re} z = x$. Let $\lambda > 0$. Then

$$C = \lambda C = \{(\lambda z_1, \lambda z_2, \lambda z_3) : (z_1, z_2, z_3) \in C\} \subset \mathbb{C}^3. \tag{5}$$

Put $S^5 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}$. Then

$$C \cap S^5 \cong T^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \tag{6}$$

where $\cong$ means a diffeomorphism.

We shall consider the following situation in the main result of this paper. Let X be a Kähler manifold of complex dimension 3 with Kähler form ω. Let Ω be a holomorphic 3-form on X such that $\Omega|_x \neq 0$ for every $x \in X$. The pair (X, Ω) is an almost Calabi–Yau manifold in the sense of Joyce [13, Definition 2.8]. Auroux [4, Example 3.3.1] gives an interesting example of special Lagrangian submanifolds with isolated singularities modelled on $C \cup -C$ in an almost Calabi–Yau manifold; see also the remark around (24).

There exists a unique C^∞ function $f : X \to \mathbb{R}_{>0}$ such that

$$f^3 \omega \wedge \omega \wedge \omega / 6 = -(\sqrt{-1}/2)^3 \Omega \wedge \overline{\Omega}. \tag{7}$$

Let g_ω be the Kähler metric of (X, ω) and put

$$g = f g_\omega. \tag{8}$$

Then Ω is a calibration on (X, g) in the sense of Harvey and Lawson by [8, III.1, Theorem 1.10].

Let M be a 3-dimensional submanifold of X.

Definition 2. M is an Ω-special Lagrangian submanifold if M is calibrated by $\operatorname{Re} \Omega$ in the sense of Harvey and Lawson, which means

$$\operatorname{Re} \Omega(e_1, e_2, e_3) = 1 \tag{9}$$

for every vector fields e_1, e_2, e_3 on M satisfying $g(e_i, e_j) = \delta_{ij}$ where g is as in (8).

(9) is equivalent to

$$\omega|_M = \operatorname{Im} \Omega|_M = 0 \tag{10}$$

by [8, III.1, Corollary 1.11]. Ω-special Lagrangian submanifolds of X are therefore Lagrangian submanifolds of (X, ω).

Suppose now that there exists a compact Ω-special Lagrangian submanifold M of X with one point singularity $x \in X$ at which M is tangent to C with multiplicity

1. C is a unique tangent cone to M at $x \in X$ by a theorem of Simon [15, Corollary, p564].

Let $a > 0$. By Theorem 1 $f^{-1}(a, 0, 0)$ is a special Lagrangian submanifold of $\mathbb{C}^3$ with singularities. Put

$$L = f^{-1}(a, 0, 0) \cap \{(z_1, z_2, z_3) : \operatorname{Re} z_1 z_2 z_3 \geq 0\} \subset \mathbb{C}^3. \tag{11}$$

Then L is a connected non-singular special Lagrangian submanifold of $\mathbb{C}^3$ diffeomorphic to $S^1 \times \mathbb{R}^2$.

Choose a compact set $K \subset \mathbb{R}^6$ sufficiently large. We may then assume that $L \setminus K$ is the graph of a normal vector field v on $C \setminus K$ with respect to the metric $|dx^1|^2 + \cdots + |dx^6|^2$. By Weinstein's theorem [21] we may regard v as a closed 1-form on $C \setminus K$. Put

$$a(L) = [v] \in H^1(C \setminus K; \mathbb{R}) \cong H^1(C \cap S^5; \mathbb{R}). \tag{12}$$

Let M be as above, and let $x \in X$ be the singular point of $M \subset X$. Regarding $M \setminus \{x\}$ as a manifold with boundary $C \cap S^5$ we have a natural homomorphism

$$f_M : H^1(M \setminus \{x\}; \mathbb{R}) \to H^1(C \cap S^5; \mathbb{R}). \tag{13}$$

Suppose

$$f_M^{-1} a(L) \neq \emptyset. \tag{14}$$

One can then glue L to M as in Joyce [13, Theorem 10.4].

Let $\mathcal{M}$ be the moduli space of compact special Lagrangian submanifolds with one point singularities where they are tangent to C with multiplicity 1; see Joyce [11, Definition 5.4] for details. By [11, p11] and [11, Corollary 6.11] there is no obstruction to the deformation of M in $\mathcal{M}$. A sufficiently small neighbourhood of M in $\mathcal{M}$ has therefore a manifold structure.

One can glue L to any M' in a sufficiently small neighbourhood of M in $\mathcal{M}$; see Joyce [13, p36] (he assumes that X is Calabi–Yau for simplicity, but one does not need it here since $Z(L) = 0$ in his notation). One can therefore define a map

$$G : (0, \epsilon) \times \mathcal{W} \to \mathcal{N} \tag{15}$$

where $\epsilon > 0$ is sufficiently small, $\mathcal{W}$ is a sufficiently small neighbourhood of M in $\mathcal{M}$, and $\mathcal{N}$ is the moduli space of compact non-singular special Lagrangian submanifolds of X. Recall that $\mathcal{N}$ is a manifold of finite dimension by a theorem of Mclean [14, Theorem 3.6].

We shall use Geometric Measure Theory in the main result of this paper. One can define locally rectifiable special Lagrangian currents; see Harvey and Lawson [8, Definition 1.3].

Theorem 3 (Harvey and Lawson [8, Theorem 4.2])**.** *Let T be a compactly supported locally rectifiable special Lagrangian current on (X, Ω). Let T' be a compactly supported 3-current on X with*

$$T' - T = \partial S \tag{16}$$

for some compactly supported 2-current on X. Then

$$\operatorname{area} T \leq \operatorname{area} T' \tag{17}$$

with respect to g defined by (8).

Corollary 4 (Harvey and Lawson)**.** *Ω-special Lagrangian submanifolds of X are minimal surfaces with respect to g defined by (8).*

We now define special Lagrangian varifolds. Let X, Ω be as above. Let T_xX be the tangent space at $x \in X$. Let $G_3(T_xX)$ be the set of all 3-dimensional $\mathbb{R}$-vector subspaces of T_xX. It is a Grassmann manifold. Put

$$G_3(TX) = \bigcup_{x\in X} G_3(T_xX), \tag{18}$$

$$G_{\mathrm{Re}\,\Omega}(TX) = \{S \in G_3(TX) : |\operatorname{Re}\Omega|_S| = 1\}. \tag{19}$$

Definition 5. An Ω-special Lagrangian varifold in X is a Radon measure on $G_{\mathrm{Re}\,\Omega}(TX)$.

Let $S \in G_{\mathrm{Re}\,\Omega}(TX)$. Choose $e_1, e_2, e_3 \in S$ such that

$$\operatorname{Re}\Omega(e_1 \wedge e_2 \wedge e_3) = 1. \tag{20}$$

Put

$$\overrightarrow{S} = e_1 \wedge e_2 \wedge e_3. \tag{21}$$

This is independent of the choice of e_1, e_2, e_3 satisfying (20).

Let V be an Ω-special Lagrangian varifold in X. Put

$$\overrightarrow{V}(\chi) = \int_{G_{\mathrm{Re}\,\Omega}(TX)} (\chi(x), \overrightarrow{S}) dV(x, S) \tag{22}$$

whenever χ is a 3-form with compact support in X. This is a current on X. Put

$$\partial\overrightarrow{V}(\chi) = \overrightarrow{V}(d\chi) \tag{23}$$

whenever χ is a 2-form with compact support in X.

Let V be a compactly supported Ω-special Lagrangian varifold in X with $\partial\overrightarrow{V} = 0$ and integer multiplicity ≥ 1. Then by Theorem 3 the first variation of V is 0 with respect to g defined by (8). It is rectifiable by a theorem of Allard [1, Theorem 5.5]; see also Simon [17, Theorem 42.4].

We now define the moduli space of special Lagrangian varifolds. Let $\mathcal{V}$ be the set of all compactly supported Ω-special Lagrangian varifolds V in X with $\partial\overrightarrow{V} = 0$ and integer multiplicity ≥ 1. We give $\mathcal{V}$ the weak topology; see Allard [1, Chapter 2.6 (2)] for the definition. It is a locally compact Hausdorff space by a theorem of Allard [1, Theorem 6.4]; see also Simon [17, Theorem 42.7 and Remark 42.8].

Let $M \subset X, \mathcal{V}$ be as above. Regarding M as a varifold in X we may write $M \in \mathcal{V}$. Let $\mathcal{N}$ be as in (15), i.e. the moduli space of compact Ω-special Lagrangian submanifolds of X. There exists a homeomorphism of $\mathcal{N}$ into $\mathcal{V}$ by Allard's regularity theorem [1, Theorem 8.19] or [17, Theorem 23.1]. We shall prove (see Corollary 11) that there exists also a homeomorphism of $\mathcal{M}$ into $\mathcal{V}$ where $\mathcal{M}$ is the moduli space of compact Ω-special Lagrangian submanifolds with one point singularities where they are tangent to C with multiplicity 1.

The main result of this paper is as follows:

Theorem 6. *There exists a neighbourhood $\mathcal{U}$ of M in $\mathcal{V}$ and a map $F : \mathcal{U} \to \mathbb{R}_{\geq 0} \times \mathcal{M}$ such that F is a homeomorphism onto its image, such that $\mathcal{U} \subset \mathcal{M} \cup \mathcal{N}$, such that $F(M') = (0, M')$ for every $M' \in \mathcal{U} \cap \mathcal{M}$ and such that $F|_{\mathcal{U}\cap\mathcal{N}}$ is a diffeomorphism onto its image whose inverse map is $G|_{F(\mathcal{U}\cap\mathcal{N})}$ where G is as in* (15).

Here is a remark on Theorem 6. If there dose not exist any M satisfying all the conditions above, the conclusion of Theorem 6 will be useless, but the proof of Theorem 6 will be useful at least in the following observation:

There exists a compact special Lagrangian submanifold M_0 with an isolated singularity modelled on $C \cup -C$ where

$$-C = \{(-z_1, -z_2, -z_3) : (z_1, z_2, z_3) \in C \subset \mathbb{C}^3\}; \tag{24}$$

see Auroux [4, Example 3.3.1]. One should therefore prove an analogue of Theorem 6 for M_0. One can use indeed the same technique as the proof of Theorem 6.

(It will not suffice however. There exist immersed special Lagrangian submanifolds $M_s, s > 0$, such that $\lim_{s\to 0} M_s = M_0$ as varifolds and such that $M_s = M_s' \cup M_s''$ for some embedded M_s', M_s'' with $M_s' \cap M_s''$ diffeomorphic to S^1. One must therefore deal with the non-isolated singularities of $M_s' \cap M_s''$.)

The remainder of this section will be devoted to giving an outline of the proof of Theorem 6. It is similar to the proof of a theorem of Donaldson [5, Theorem 11] for anti-self-dual Yang–Mills connections. We give first a review of Donaldson's theorem.

Consider anti-self-dual Yang–Mills connections on an SU_2 bundle with $c_2 = 1$ over a compact simply connected oriented Riemannian manifold Z of dimension 4 with $b_+^2 = 0$. There exists an anti-self-dual Yang–Mills connection A_0 over $\mathbb{R}^4$ whose curvature $F(A_0)$ satisfies $\int_{\mathbb{R}^4} |F(A_0)|^2 = 8\pi^2$ and is close to the δ function at $0 \in \mathbb{R}^4$ as a Radon measure on $\mathbb{R}^4$. Let $s > 0$ be sufficiently small. Dilating A_0 by $s > 0$ one obtains $s_\# A_0$, whose curvature tends as $s \to 0$ to the δ function at $0 \in \mathbb{R}^4$ as a Radon measure on $\mathbb{R}^4$. One can glue $s_\# A_0$ at any $z \in Z$ to the connection of curvature 0 over Z; see Taubes [18]. One then obtains an anti-self-dual Yang–Mills connection $A_{s,z}$ over Z. The curvature of $A_{s,z}$ tends as $s \to 0$ to the δ function at $z \in Z$ as a Radon measure on Z.

Donaldson [5, Theorem 11] proved surjectivity of Taubes' gluing; see also Freed and Uhlenbeck [7, Section 9].

Theorem 7 (Donaldson)**.** *If an anti-self-dual Yang–Mills connection has curvature sufficiently close to a δ-function as a Radon measure on Z, its gauge equivalence class is equal to $[A_{s,z}]$ for some $s > 0, z \in Z$.*

We give here an outline of the proof of Theorem 7. The first step is a bubbling-off analysis. Letting F_A be the curvature of A one has $\int_Z |F_A|^2 = 8\pi^2$ since A is anti-self-dual with $c_2 = 1$. Given $z \in Z$ one has the smallest ball $B(z) \subset Z$ centred at z such that $\int_{B(z)} |F_A|^2 = 7\pi^2$. Choose $w \in Z$ such that the radius of $B(w)$ attains the minimum. Restricting A to a neighbourhood of w one can define $r_\#^{-1} A$, where r is the radius of $B(w)$. By Uhlenbeck's a-priori estimate [20] $r_\#^{-1} A$ tends to some anti-self-dual Yang–Mills connection A' over $T_w Z \cong \mathbb{R}^4$ up to gauge equivalence as F_A tends to the δ funtion at w; i.e. A' has bubbled off. By Atiyah, Hitchin and Singer's theorem [3, Theorem 9.1] its gauge equivalence class $[A']$ is equal to $[A_0]$ up to dilation and translation on $\mathbb{R}^4$. Dilating A_0 suitably, one can assume $[A'] = [A_0]$. Hence $[A]$ is sufficiently close to $[r_\# A'] = [r_\# A_0]$ over some ball B_1 centred at w. By Uhlenbeck's a-priori estimate [20] A is close to the connection of curvature 0 over $Z \setminus B_2$ for some $B_2 \supset B_1$. One has not seen yet, however, what $[A]$ looks like over $B_2 \setminus B_1$. One uses here again Uhlenbeck's a-priori estimate [20]. One then obtains $[A] = [A_{r,w}]$ by gluing $r_\# A_0$ at w to the connection of curvature 0 over Z. This is an outline of the proof of Theorem 7.

Consider now the moduli space of all gauge equivalence classes of anti-self-dual Yang-Mills connections over Z. One can compactify the moduli space by adding the ideal connections whose curvatures are δ functions on Z. One sees from Theorem 7 that $(s, z) \mapsto [A_{s,z}]$ parametrizes a neighbourhood of the boundary of the moduli space.

We give now an outline of the proof of Theorem 6 making an analogy with Theorem 7. We first prove an analogue of Uhlenbeck's removable singularities theorem [20]. The idea of Uhlenbeck's theorem is as follows. Let A be an anti-self-dual Yang–Mills connection over $(T, \infty) \times S^3$ with $T > 0$ sufficiently large. $\{A|_{\{t\}\times S^3} : t \in (T, \infty)\}$ is the gradient flow of the Chern–Simons functional on the

space of connections over S^3. If the curvature of A is sufficiently small, the gradient flow converges.

By Simon's technique [15, 16] minimal surfaces look like a gradient flow near isolated singularities with multiplicity 1 smooth tangent cones. We shall define an energy of such minimal surfaces and prove that Simon's gradient-like flow converges if the minimal surface has sufficiently small energy; see Theorem 9 for a more precise statement. It is an analogue of Uhlenbeck's removable singularities theorem [20].

We next prove an analogue of Atiyah, Hitchin and Singer's theorem [3, Theorem 9.1]. What should bubble off in our case is a special Lagrangian varifold in $\mathbb{R}^6$ tangent to C with multiplicity 1 at infinity in $\mathbb{R}^6$. We shall prove a uniqueness theorem for such varifolds. More precisely:

Theorem 8. *Let V be a special Lagrangian varifold with $\partial\overrightarrow{V} = 0$ in $\mathbb{R}^6$ tangent to C with multiplicity 1 at infinity in $\mathbb{R}^6$. Then V is represented with multiplicity 1 by C or there exist $a \in \mathbb{R}$ such that V is represented with multiplicity 1 by L up to the action of $SU_3 \ltimes \mathbb{C}^3$ where L is as in (11); notice here that L depends on $a \in \mathbb{R}$.*

It will be important in the proof of Theorem 8 that C has symmetry; i.e. there exists a subgroup of SU_3 whose action preserves $C \subset \mathbb{R}^6$. This is the main difference from the previous paper [9], where we have considered singularities modelled on the transverse intersection of two special Lagrangian planes, but it does not have such symmetry in general.

As one uses Uhlenbeck's technique again in the final step to the proof of Theorem 7, so we use Simon's technique again in the final step to the proof of Theorem 6. This is an outline of the proof of Theorem 6.

In Section 2 we prove the analogue of Uhlenbeck's removable singularities theorem. In Section 3 we apply the result of Section 2 to the bubbling off analysis. In Section 4 we prove the analogue of Atiyah, Hitchin and Singer's theorem. In Section 5 we complete the proof of Theorem 6.

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2. Analogue of Uhlenbeck's Removable Singularities Theorem

In this section we prove an analogue of Uhlenbeck's removable singularities theorem [20, Theorem 4.1]; see Section 1 for the analogy between them.

Let $G_3(\mathbb{R}^6)$ be the set of all 3-dimensional vector subspaces of $\mathbb{R}^6$. It is a Grassmannian manifold. Let V be a varifold of dimension 3 in $\mathbb{R}^6$, i.e. a Radon measure on $\mathbb{R}^6 \times G_3(\mathbb{R}^6)$. Put

$$E(V) = \int_{\mathbb{R}^6 \times G_3(\mathbb{R}^6)} |x|^{-5} |S^\perp x|^2 dV(x, S), \tag{25}$$

where $|x| = \sqrt{x_1^2 + \cdots + x_6^2}$ for each $x = (x_1, \ldots, x_6)$ and $S^\perp x$ is obtained by projecting x onto $S^\perp$, which is the orthogonal complement of the vector subspace S with respect to the metric $|dx^1|^2 + \cdots + |dx^6|^2$ on $\mathbb{R}^6$. This is the definition of the energy of V. One uses (25) in the monotonicity formula [17, Theorem 17.6].

Let $r > 0$. Put

$$B_r = \{x \in \mathbb{R}^6 : |x| < r\}, \tag{26}$$

where $|x| = \sqrt{x_1^2 + \cdots + x_6^2}$ for each $x = (x_1, \ldots, x_6)$. Let $s \in (0, r)$. Put

$$A_{s,r} = B_r \setminus \overline{B_s}, \tag{27}$$

$$g_{\text{cyl}} = (|dx^1|^2 + \cdots + |dx^6|^2)/|x|^2; \tag{28}$$

g_{cyl} is a Riemannian metric on $A_{r,s}$. Let $\|v\|_{C^2_{\mathrm{cyl}}}$ be the C^2 norm induced from g_{cyl}, where v is any normal vector field on $C \cap A_{s,r}$ with respect to g_{cyl}. Let $\mathrm{graph}_{\mathrm{cyl}}\, v$ be its graph, which is a submanifold of $A_{s,r}$ if $\|v\|_{C^0_{\mathrm{cyl}}}$ is sufficiently small.

Let V be a varifold of dimension 3 in $A_{s,r}$. Let C be as in (4). Define $d(V, C \cap A_{s,r}) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ as follows: put

$$d(V, C \cap A_{s,r}) = \|v\|_{C^2_{\mathrm{cyl}}(C \cap A_{s,r})} \tag{29}$$

if V is represented by $\mathrm{graph}_{\mathrm{cyl}}\, v$ with multiplicity 1 for some normal vector field v on $C \cap A_{s,r}$ with respect to g_{cyl}; put

$$d(V, C \cap A_{s,r}) = \infty \tag{30}$$

if there does not exist any such v.

Notice that SU_3 is the group of $\mathbb{C}$-vector space isomorphisms of $\mathbb{C}^3 = \mathbb{R}^6$ preserving

$$g_{\mathbb{R}^6} = |dx^1|^2 + \cdots + |dx^6|^2, \tag{31}$$

$$\Omega_{\mathbb{R}^6} = (dx^1 + \sqrt{-1}dx^2) \wedge (dx^3 + \sqrt{-1}dx^4) \wedge (dx^5 + \sqrt{-1}dx^6). \tag{32}$$

The following theorem is an analogue of Uhlenbeck's removable singularities theorem; see the discussion in Section 1 for the analogy between them.

Theorem 9. *There exists $\epsilon_9 > 0$ such that the following holds:*

Let $r, \epsilon \in (0, \epsilon_9)$. Let g be a Riemannian metric on B_r with

$$g = g_{\mathbb{R}^6} + O(|x|^2). \tag{33}$$

Let Ω be a complex volume form on (B_r, g) with

$$\Omega = \Omega_{\mathbb{R}^6} + O(|x|). \tag{34}$$

Let V be an Ω-special Lagrangian varifold with $\partial\overrightarrow{V} = 0$ in (B_r, g). Suppose

$$\max\{E(V), d(V, C \cap A_{r/2,r})\} < \epsilon. \tag{35}$$

Then there exist $\lambda \in (0,1), \gamma \in SU_3$ with $\mathrm{dist}(1, \gamma) < \epsilon$ such that

$$d(V, \gamma(C) \cap A_{\lambda^{k+1}, \lambda^k}) < \epsilon 2^{-k} \tag{36}$$

for any integer $k \geq 0$.

Once we have proved Theorem 9 we shall obtain:

Corollary 10. *Let V be an Ω-special Lagrangian varifold with $\partial\overrightarrow{V} = 0$ in a Calabi–Yau manifold X. Suppose that V is sufficiently close to $M \in \mathcal{M}$ as a varifold, where $\mathcal{M}$ is the moduli space of compact Ω-special Lagrangian submanifolds with one point singularities modelled on C. Let $x \in X$ and choose normal coordinates at x with respect to g defined by* (8)*. Suppose that V restricted to a neighbourhood of x has energy sufficiently small. Then $V \in \mathcal{M}$ and it is arbitrarily close to M in the topology of $\mathcal{M}$ given by Joyce* [11, Definition 5.6].

Here is another corollary of Theorem 9. It has been used in the statement of Theorem 6.

Corollary 11. *There exists a homeomorphism of $\mathcal{M}$ into $\mathcal{V}$ where $\mathcal{M}$ is as above and $\mathcal{V}$ is the moduli space of compactly supported Ω-special Lagrangian varifolds V with $\partial\overrightarrow{V} = 0$ and integer multiplicity ≥ 1.*

We shall here prove Corollary 11 assuming Corollary 10. Recall that Corollary 10 will be obtained immediately once Theorem 9 has been proved.

Proof of Theorem 11. Define $i_{\mathcal{M}} : \mathcal{M} \to \mathcal{V}$ by

$$i_{\mathcal{M}}(M') : f \mapsto \int_{M'} f(x, T_x M') d\|M'\|(x) \tag{37}$$

for every compactly supported continuous function f on $G_3(TX)$ where $\|M'\|$ is the volume form of M' induced from g defined by (8). We shall prove that it is a homeomorphism. It is clearly one-to-one and continuous. It will therefore suffice to prove that $i_{\mathcal{M}}$ is an open mapping. Let $M \in \mathcal{M}$. Let x be the singular point of M. Choose normal coordinates at x with respect to g defined by (8). Given a varifold V in X we define its local energy by first restricting V to a neighbourhood of x and then applying (25). We shall here use Corollary 10 to prove Corollary 11. It will then suffice to prove that if $M' \in \mathcal{M}$ is sufficiently close to M as a varifold, the local energy of M' is arbitrarily small. We may here assume that $M = C$ and that M' is tangent to C with multiplicity 1 at 0 in $\mathbb{R}^6$. By Lemma 16 M' has bounded mean curvature with respect to the metric $|dx^1|^2 + \cdots + |dx^6|^2$, where $x^1, \ldots, x^6$ are the normal coordinates which we have chosen above. Hence by the monotonicity formula [17, Theorem 17.6] there exists a constant $k > 0$ such that

$$E(M' \cap B_r \setminus \overline{B_s}) \leq e^{kr} r^{-3} \operatorname{area}(M' \cap B_r) - e^{ks} s^{-3} \operatorname{area}(M' \cap B_s), \tag{38}$$

where $E(\bullet), B_r$ are as in (25), (26) respectively. Since M' is tangent to C with multiplicity 1 at 0 in $\mathbb{R}^6$ we have

$$\lim_{s \to 0} e^{ks} s^{-3} \operatorname{area}(M' \cap B_s) = \operatorname{area}(C \cap B_1). \tag{39}$$

By (38) and (39) we have

$$E(M' \cap B_r) \leq e^{kr} r^{-3} \operatorname{area}(M' \cap B_r) - \operatorname{area}(C \cap B_1). \tag{40}$$

By (40) the local energy of M' is arbitrarily small if $r > 0$ is sufficiently small and M' is sufficiently close to $M = C$ as a varifold. This completes the proof of Theorem 11. □

The remainder of this section will be devoted to a proof of Theorem 9. We first prove:

Lemma 12. *There exist $\eta, k > 0$ such that the following holds:*

Let g be a Riemannian metric on $A_{1/4,2}$ with $\|g - g_{\mathbb{R}^6}\|_{C^1} < \eta$. Let Ω be a complex volume form on $(A_{1/4,2}, g)$ with $\|\Omega - \Omega_{\mathbb{R}^6}\|_{C^0} < \eta$. Let V be an Ω-special Lagrangian varifold in $A_{1/4,2}$. Suppose

$$d(V, C \cap A_{1/4,2}) < \eta. \tag{41}$$

Then

$$\inf_{\gamma \in SU_3} d(V, \gamma(C) \cap A_{1/2,1}) \leq k \max\{E(V), \|g_{\mathbb{R}^6} - g\|_{C^1}, \|\Omega_{\mathbb{R}^6} - \Omega\|_{C^0}\}. \tag{42}$$

Proof. Suppose that Lemma 12 is false. Then for any $i = 1, 2, \ldots$ there exists a Riemannian metric g_i on $A_{1/4,2}$, a complex volume form Ω_i on $(A_{1/4,2}, g_i)$ and a special Lagrangian varifold V_i in $(A_{1/4,2}, \Omega_i)$ such that

$$d(V_i, C \cap A_{1/4,2}) < 1/i, \tag{43}$$

$$\inf_{\gamma \in SU_3} d(V_i, \gamma(C) \cap A_{1/2,1}) > i \max\{E(V_i), \|g_{\mathbb{R}^6} - g_i\|_{C^1}, \|\Omega_{\mathbb{R}^6} - \Omega_i\|_{C^0}\}. \tag{44}$$

It will suffice to obtain a contradiction from (43) and (44). For any $i = 1, 2, \ldots$ there exists $\gamma_i \in SU_3$ such that

$$d(V_i, \gamma_i(C) \cap A_{1/2,1}) = \inf_{\gamma \in SU_3} d(V_i, \gamma(C) \cap A_{1/2,1}). \tag{45}$$

By (43) we have

$$\lim_{i\to\infty} \gamma_i = 1, \tag{46}$$

$$\lim_{i\to\infty} d(V_i, \gamma_i(C)\cap A_{1/4,2}) = 0. \tag{47}$$

There exists a normal vector field v_i on $C \cap A_{1/4,2}$ such that

$$d(V_i, \gamma_i(C)\cap A_{1/2,1}) = \|v_i\|_{C^2_{\rm cyl}}. \tag{48}$$

$\gamma_{i\#}^{-1} V_i$ is represented with multiplicity 1 by $\mathrm{graph}_{\rm cyl}\, v_i$. By (29) and (44) we have

$$\|v_i\|_{C^2_{\rm cyl}(C\cap A_{1/2,1})} > i \max\{E(V_i), \|g_{\mathbb{R}^6} - g_i\|_{C^1}, \|\Omega_{\mathbb{R}^6} - \Omega_i\|_{C^0}\}. \tag{49}$$

By (29) and (47) we have

$$\lim_{i\to\infty} \|v_i\|_{C^2_{\rm cyl}(C\cap A_{1/4,2})} = 0. \tag{50}$$

By [15, p561, (7.13)] we have

$$\|\partial_t v_i\|_{L^2_{\rm cyl}(C\cap A_{1/4,2})} < 2E(V_i), \tag{51}$$

where $\partial_t = \partial/\partial t$ and $t = -\log|z|$. Hence there exists a constant $k > 0$ such that

$$\|v_i\|_{L^2_{\rm cyl}(C\cap A_{1/4,2})} \le \|v_i\|_{L^2_{\rm cyl}(C\cap A_{1/2,1})} + kE(V_i). \tag{52}$$

Put $u_i = v_i / \|v_i\|_{C^2_{\rm cyl}(C\cap A_{1/2,1})}$. By (49), (50), (52) and elliptic regularity

$$\sup_{i=1,2,\dots} \|u_i\|_{C^2_{\rm cyl}(K)} < \infty \tag{53}$$

for each compact $K \subset C \cap A_{1/4,2}$. Hence there exists a subsequence $(u_{i_j})_{j=1}^\infty$ and its limit

$$\lim_{j\to\infty} u_{i_j} = u_\infty \in C^2_{\rm cyl}(C \cap A_{1/4,2}); \tag{54}$$

this is the limit in the local C^2 topology. By (49) and (50) u_∞ is a special Lagrangian Jacobi field on $C\cap A_{1/4,2}$. By (49) and (51) we have $\partial_t u_\infty = 0$. Hence u_∞ is a special Legendrian Jacobi field on $C \cap S^5$; see [10, Definition 6.7]. By [11, p11] we may regard u_∞ as an element of $\mathfrak{su}_3 = T_1 SU_3$. Put

$$\delta_j = \|v_{i_j}\|_{C^2_{\rm cyl}(C\cap A_{1/2,1})}. \tag{55}$$

By (50) we have $\delta_j \to 0$ as $j \to \infty$. By (54) we have

$$v_{i_j} - \delta_j u_\infty = o(\delta_j). \tag{56}$$

Putting $\beta_j = \exp \delta_j u_\infty \in SU_3$ we have

$$d(V_{i_j}, \beta_{i_j}\gamma_{i_j}(C)) = o(\delta_j). \tag{57}$$

By (45) $\inf_{\gamma\in SU_3} d(V_{i_j}, \gamma(C)\cap A_{1/2,1}) = 0$. This contradicts (44). We have therefore completed the proof of Lemma 12. □

We shall use:

Lemma 13 ([9, Lemma 3.6]). *For any $\eta > 0$ there exists $\epsilon_{13} > 0$ such that the following holds:*

Let g be a Riemannian metric on B_1 with $\|g - g_{\mathbb{R}^6}\|_{C^1} < \epsilon_{13}$. Let Ω be a complex volume form on (B_1, g) with $\|\Omega - \Omega_{\mathbb{R}^6}\|_{C^1} < \epsilon_{13}$. Let V be an Ω-special Lagrangian varifold with $\partial \overrightarrow{V} = 0$ in (B_1, g). Suppose

$$\max\{d(V, C\cap A_{1/2,1}), E(V \cap A_{1/8,1})\} < \epsilon_{13}. \tag{58}$$

Then

$$d(V, C\cap A_{1/4,1}) < \eta. \tag{59}$$

From Lemmata 13 and 12 we obtain:

Corollary 14. *There exists $\epsilon > 0$ such that the following holds:*
Let r, g, Ω, V be as in Theorem 9. Then for all $i = 1, 2, \dots$ we have

$$d(V, C \cap A_{r/2^i, r/2^{i-1}}) < \epsilon. \tag{60}$$

We next prove:

Lemma 15. *There exists $\epsilon > 0$ such that the following holds:*
Let r, g, Ω, V be as in Theorem 9. Then

$$\lim_{s \to 0} \|V\|(B_s)/s^3 = \|C\|(B_1). \tag{61}$$

Proof. By Corollary 14 and Lemma 16 we may use the the monotonicity formula [17, Theorem 17.6]; i.e. there exists a constant $k > 0$ such that $e^{kr}\|V\|(B_r)/r^3$ is a monotone decreasing function of $r > 0$. Hence $\lim_{r\to 0} \|V\|(B_r)/r^3$ exists. It will therefore suffice to find $r(1) > r(2) > \dots$ tending to 0 such that

$$\lim_{i \to \infty} \|V\|(B_{r(i)})/r(i)^3 = \|C\|(B_1). \tag{62}$$

By Corollary 14 there exist $r(1), r(2), \dots > 0$ tending to 0 and $\gamma_1, \gamma_2, \dots \in SU_3$ tending to some $\gamma \in SU_3$ such that $r(i)^{-1}_{\#}\gamma^{-1}_{i\#}V$ restricted to $A_{1/2,1}$ is represented with multiplicity 1 by $\operatorname{graph}_{\mathrm{cyl}} v_i$ for some normal vector field v_i on $C \cap A_{1/2,1}$ and such that $(v_i)_{i=1,2,\dots}$ converges to some v in the local C^2 topology. We have

$$E(V) = \sum_{i=1}^{\infty} E(\operatorname{graph}_{\mathrm{cyl}} v_i) < \infty. \tag{63}$$

Hence $E(\operatorname{graph}_{\mathrm{cyl}} v_i) \to 0$ as $i \to \infty$. Arguing as in the proof of Lemma 12 one finds $\gamma \in SU_3$ such that

$$\lim_{i \to \infty} r(i)^{-1}(\operatorname{spt}\|V\| \cap \partial B_{r(i)}) = \gamma(C) \cap \partial B_1 \tag{64}$$

in the C^2 topology. We shall now use the technique in [9, Section 3]. Put

$$\Psi_{S^5} = ((x^1\partial_1 + \dots + x^6\partial_6) \lrcorner \Omega_{\mathbb{R}^6})|_{S^5}. \tag{65}$$

By [9, Proposition 3.3] and the proof of [9, Proposition 3.4] there exists a 2-form Ψ on $\mathbb{R}^6 \setminus \{0\}$ such that

$$\operatorname{Re}\Omega = d(3^{-1}|x|^3\Psi), \tag{66}$$

$$|\Psi - \Psi_{S^5}|_{\mathrm{cyl}} \le |\Omega - \Omega_{\mathbb{R}^6}|. \tag{67}$$

By (66) we have

$$r^{-3}\|V\|(B_r) = r^{-3}\int_{\operatorname{spt}\|V\| \cap B_r} d(3^{-1}|x|^3\Psi) \tag{68}$$

$$= \int_{r^{-1}(\operatorname{spt}\|V\| \cap \partial B_r)} 3^{-1}\Psi|_{\partial B_r} \tag{69}$$

$$= \int_{r^{-1}(\operatorname{spt}\|V\| \cap \partial B_r)} 3^{-1}(\Psi|_{\partial B_r} - \Psi_{S^5} + \Psi_{S^5}). \tag{70}$$

By (67) we have

$$\left|\int_{r^{-1}(\operatorname{spt}\|V\| \cap \partial B_r)} 3^{-1}(\Psi|_{\partial B_r} - \Psi_{S^5})\right| \tag{71}$$

$$\le \operatorname{area}(r^{-1}(\operatorname{spt}\|V\| \cap \partial B_r))3^{-1}|\Psi|_{\partial B_r} - \Psi_{S^5}|_{\mathrm{cyl}} \tag{72}$$

$$\le k \sup_{B_r} |\Omega - \Omega_{\mathbb{R}^6}| \tag{73}$$

for some constant $k > 0$. By assumption

$$\lim_{r\to 0} \sup_{B_r} |\Omega - \Omega_{\mathbb{R}^6}| = 0. \tag{74}$$

By (71)–(74) we have

$$\lim_{r\to 0} \int_{r^{-1}(\operatorname{spt}\|V\|\cap \partial B_r)} 3^{-1}(\Psi|_{\partial B_r} - \Psi_{S^5}) = 0. \tag{75}$$

By (64) we have

$$\lim_{i\to\infty} \int_{r(i)^{-1}(\operatorname{spt}\|V\|\cap \partial B_{r(i)})} 3^{-1}\Psi_{S^5} = \int_{\gamma(C)\cap\partial B_1} \Psi_{S^5}. \tag{76}$$

By definition

$$\int_{\gamma(C)\cap\partial B_1} \Psi_{S^5} = \operatorname{area}(\gamma(C)\cap B_1) = \operatorname{area}(C\cap S^5). \tag{77}$$

From (68)–(70), (75) and (76) we obtain (62). Hence (61) holds, which completes the proof of Lemma 15. □

We shall need the following lemma. One can prove it by calculation.

Lemma 16. *Let V be a varifold of first variation 0 in B_r with respect to a Riemannian metric g with*

$$g = g_{\mathbb{R}^6} + O(|x|^2). \tag{78}$$

Suppose

$$d(V, C\cap A_{r/2,r}) < \epsilon. \tag{79}$$

Let $H_{r/2,r}$ be the mean curvature of $\operatorname{spt}\|V\|\cap A_{r/2,r}$ *with respect to $g_{\mathbb{R}^6}$. Then there exists $k > 0$ depending only on g, ϵ such that*

$$|H_{r/2,r}| \le k|x|. \tag{80}$$

By Lemma 16 one can use the monotonicity formula [17, Theorem 17.6]. Moreover by Lemma 15 one can proceed as in the proof of [15, p560, (7.8)]. One then obtains:

Lemma 17. *There exist $\epsilon, k > 0$ such that the following holds:*

Let r, g, Ω, V be as in Theorem 9. Let $s \in (0, r)$. Then $\operatorname{spt}\|V\|\cap\partial B_s$ is a submanifold of ∂B_s and

$$s^{-2}\operatorname{area}(\operatorname{spt}\|V\|\cap\partial B_s) \ge E(V) + \operatorname{area}(C\cap S^5) - ks. \tag{81}$$

We prove finally the following lemma arguing as in the proof of [16, Lemma 6.4]. We use here again the fact that C is special Lagrangian Jacobi integrable; see Joyce [11, p11].

Lemma 18. *There exist $\epsilon, \lambda, q > 0$ such that the following holds:*

Let g be a Riemannian metric on B_1 with $\|g - g_{\mathbb{R}^6}\|_{C^1} < \epsilon$. Let Ω be a complex volume form on B_1 with $\|\Omega - \Omega_{\mathbb{R}^6}\|_{C^0} < \epsilon$. Let V be an Ω-special Lagrangian varifold with $\partial\overrightarrow{V} = 0$ in (B_1, g). Suppose

$$d(V, C\cap A_{\lambda^2,1}) < \epsilon. \tag{82}$$

Let $k > 0$. Suppose that for any $s \in (0,1)$, $\operatorname{spt}\|V\|\cap\partial B_s$ is a submanifold of ∂B_s and

$$s^{-2}\operatorname{area}(\operatorname{spt}\|V\|\cap\partial B_s) \ge E(V) + \operatorname{area}(C\cap S^5) - ks. \tag{83}$$

Then

$$\inf_{\gamma\in SU_3} d(V, \gamma(C)\cap A_{\lambda^2,\lambda}) \le (1/2)\max\{\inf_{\operatorname{dist}(1,\gamma)<\epsilon} d(V, \gamma(C)\cap A_{\lambda,1}), qk\}. \tag{84}$$

Lemma 18 will be proved in a similar way to [16, Lemma 6.4].

Proof of Lemma 18. Suppose that Lemma 18 is false. Then there exist $\epsilon_1 > \epsilon_2 > \ldots$ tending to 0, $V_1, V_2, \ldots, k_1, k_2, \ldots$ as above and $\gamma_1, \gamma_2, \ldots$ with $\operatorname{dist}(1, \gamma_i) < \epsilon_i$ such that

$$\inf_{\gamma \in SU_3} d(V_i, \gamma(C) \cap A_{\lambda^2,\lambda}) > (1/2) \max\{d(V_i, \gamma_i(C) \cap A_{\lambda,1}), ik_i\} \tag{85}$$

for all integer $i \geq 1$. By (82) we have

$$\inf_{\gamma \in SU_3} d(V_i, \gamma(C) \cap A_{\lambda^2,\lambda}) < \infty. \tag{86}$$

By (85) and (86) we have $k_i \to 0$. Hence by (83) and Lemma 13 there exists a normal vector field v_i on $C \cap A_{\delta_i,1}$ such that V_i restricted to $A_{\delta_i,1}$ is represented with multiplicity 1 by $\operatorname{graph}_{\mathrm{cyl}} v_i$ with $\delta_i \to 0$ as $i \to \infty$. Put $T_i = -\log \delta_i$. Put $t = -\log |x|$. By [15, p561, (7.13)] we have

$$\|\partial_t v_i\|_{L^2_{\mathrm{cyl}}(C \cap A_{\delta_i,1})} < 2E(V_i). \tag{87}$$

By (87), (83) and elliptic regularity

$$\int_0^{T_i} \|\partial_t v_i\|^2_{L^2_{\mathrm{cyl}}(C \cap S^5)} dt \leq q(\|v_i\|^2_{L^2_{\mathrm{cyl}}(C \cap A_{\lambda,1})} + k_i) \tag{88}$$

for some constant $q > 0$ independent of i. Let $r \geq \delta_i$. By (88) we have

$$\|v_i\|_{C^2_{\mathrm{cyl}}(C \cap A_{r,1})} \leq q_r(\|v_i\|_{C^2_{\mathrm{cyl}}(C \cap A_{\lambda,1})} + k_i) \tag{89}$$

for some $q_r > 0$ independent of i. By (85) and (89) we have

$$\|v_i\|_{C^2_{\mathrm{cyl}}(C \cap A_{\lambda,1})} \geq k_i. \tag{90}$$

Put $u_i = v_i / \|v_i\|_{C^2_{\mathrm{cyl}}(C \cap A_{\lambda,1})}$. By (89) and (90) we have

$$\sup_{i=1,2,\ldots} \|u_i\|_{C^2_{\mathrm{cyl}}(C \cap A_{\lambda,1})} < \infty. \tag{91}$$

Hence there exists a subsequence $(u_{i_j})_{j=1,2,\ldots}$ such that $u_{i_j} \to w$ as $j \to \infty$ in the local C^∞ topology for some $w \in C^\infty(C \cap A_{0,1})$. By (88) and (90) we have

$$\int_0^\infty \|\partial_t w\|^2_{L^2(C \cap S^5)} < \infty. \tag{92}$$

One can therefore find a basis $(b_l)_{l \in \mathbb{Z}}$ of $L^2(C \cap S^5)$ with

$$w = w_0 + \sum_{l=1}^\infty \operatorname{Re}((a_l + a_l' t) e^{-\alpha_l t}) b_l \tag{93}$$

for some w_0, a_l, a_l', α_l such that $\operatorname{Re} \alpha_l > 0$ and such that w_0 is a special Legendrian Jacobi field on $C \cap S^5$; see [10, Definition 6.7]. Put $\beta_i = \|v_i\|_{C^2_{\mathrm{cyl}}(C \cap A_{\lambda,1})}$. Put $w_+ = w - w_0$. Then

$$v_i = \beta_i(w_0 + w_+) + o(\beta_i). \tag{94}$$

Since C is stable in the sense of Joyce [11, p11] we may regard w_0 as an element of the Lie algebra $T_1 SU_3$. One can therefore find, using (94), $\sigma_i \in SU_3$ such that $\sigma_{i\#}^{-1} V_i$ is represented with multiplicity 1 by $\operatorname{graph}_{\mathrm{cyl}} z_i$ for some normal vector field z_i on $C \cap A_{\delta_i,1}$ with

$$z_i = \beta_i w_+ + o(\beta_i). \tag{95}$$

Now choose $\lambda > 0$ sufficiently large. Then, since $\operatorname{Re} \alpha_l > 0$,

$$\|w_+\|_{C^2(C \cap A_{\lambda^2,\lambda})} \leq (1/100) \|w_+\|_{C^2(C \cap A_{\lambda,1})}. \tag{96}$$

By (95) and (96) we have

$$\|z_i\|_{C^2(C\cap A_{\lambda^2,\lambda})} \le (1/10)\|z_i\|_{C^2(C\cap A_{\lambda,1})}. \tag{97}$$

This contradicts (85), which completes the proof of Lemma 18. $\square$

Let r, g, Ω, V be as in Theorem 9. Using Lemmata 17 and 18 repeatedly one can find $\gamma \in SU_3$ with $\mathrm{dist}(1,\gamma) < \epsilon$ such that

$$d(V, \gamma(C) \cap A_{r\lambda^{i+1}, r\lambda^i}) < \epsilon 2^{-i} \tag{98}$$

for any integer $i \ge 0$. This completes the proof of Theorem 9.

3. Bubbling-off

In this section we prove Theorem 19 below. We shall continue to use (25), (29) and some other notation in the previous section.

Theorem 19. *There exist $\epsilon_{19}, \eta > 0$ such that the following holds:*
Let $r \in (0, \epsilon_{19})$. Let g be a Riemannian metric on B_r with $g = g_{\mathbb{R}^6} + O(|x|^2)$. Let Ω be a complex volume form on B_r with $\Omega = \Omega_{\mathbb{R}^6} + O(|x|)$. Let V be an Ω-special Lagrangian varifold with $\partial\overrightarrow{V} = 0$ in (B_r, g). Suppose that V is tangent to C with multiplicity 1 at $0 \in B_r$. Suppose

$$E(V) < \epsilon_{19}/2, \tag{99}$$

$$d(C, V \cap A_{r/2,r}) < \eta/2. \tag{100}$$

Let $(V_i)_{i=1,2,\dots}$ be a sequence of Ω-special Lagrangian varifolds with $\partial\overrightarrow{V_i} = 0$ in (B_r, g). Suppose that $(V_i)_{i=1,2,\dots}$ converges to V. Then there exists a subsequence $(V_{i_j})_{j=1,2,\dots}$ such that either of the following statements holds: for all $j = 1, 2, \dots$ there exist $b_j \in B_r, \gamma_j \in SU_3$ such that V_{i_j} is tangent to $\gamma_j(C)$ with multiplicity 1 at b_j; or there exist $b_1, b_2, \dots \in B_r$ tending to 0 and $\delta_1, \delta_2, \dots > 0$ tending to 0 such that $\{\delta_{j\#}^{-1}(V_{i_j} - b_j)\}_{j=1,2,\dots}$ converges to some $\Omega_{\mathbb{R}^6}$-special Lagrangian varifold W in $(\mathbb{R}^6, g_{\mathbb{R}^6})$ such that W is tangent to $\gamma(C)$ with multiplicity 1 at infinity in $\mathbb{R}^6$ for some $\gamma \in SU_3$ and such that $E(W - a) \ge \epsilon_{19}/100$ for any $a \in \mathbb{R}^6$, where $W - a$ is the translate of W by a on $\mathbb{R}^6$.

We first prove the following theorem using techniques in the previous paper [9]:

Theorem 20. *There exist $\epsilon, k > 0$ such that the following holds:*
Let $r > 0$. Let g, Ω, V be as in Theorem 19. Suppose $d(V, C \cap A_{r/2,r}) < \eta$. Suppose $E(V \cap A_{s,r}) < \epsilon$ for some $s \in [0, r/2)$. Then $d(V, C \cap A_{s,r}) < \eta^k$.

Proof of Theorem 20. From Lemma 12 we obtain:

Lemma 21. *There exist $\eta, k > 0$ such that the following holds:*
Let g be a Riemannian metric on $A_{1/4,2}$ with $\|g - g_{\mathbb{R}^6}\|_{C^1} < \eta$. Let Ω be a complex volume form on $A_{1/4.2}$ with $\|\Omega - \Omega_{\mathbb{R}^6}\|_{C^0} < \eta$. Let V be an Ω-special Lagrangian varifold in $(A_{1/4,2}, g)$. Suppose $d(V, C \cap A_{1/4,2}) < \eta$. Then

$$\sup_{s\in(1/2,1)} |\operatorname{area}(s^{-1}(\operatorname{spt}\|V\| \cap \partial B_s)) - \operatorname{area}(C \cap S^5)| \tag{101}$$

$$\le k \max\{E(V), \|g_{\mathbb{R}^6} - g\|_{C^1}, \|\Omega_{\mathbb{R}^6} - \Omega\|_{C^0}\}. \tag{102}$$

Using this lemma we shall argue as in the proof of [9, Proposition 5.1]. Suppose that there exists $q \in (s, r/2)$ such that $d(V, C \cap A_{q,r}) < \eta$. By Lemma 21 we may assume

$$\sup_{p\in(2q,r)} |\operatorname{area}(p^{-1}(V \cap \partial B_p)) - \operatorname{area}(C \cap S^5)| < \eta. \tag{103}$$

By (103) and [9, Lemma 4.3] there exists $k > 0$ such that $d(V, C \cap A_{2q,r}) \leq \eta^k$. One can therefore prove $d(V, C \cap A_{s,r}) < \eta^k$ using Lemma 13. This completes the proof of Theorem 20. $\square$

Theorem 20 will be used in the following:

Proof of Theorem 19. By Theorem 9 it will suffice to find W as in Theorem 19 supposing that $E(V_i - b) \geq \epsilon$ for any integer $i \geq 1$ and $b \in B_{r\epsilon}$. Let i be any integer ≥ 1. Put

$$\delta_i(b) = \min\{\delta > 0 : E(V_i - b \cap A_{\delta,r}) = \epsilon/2\} \tag{104}$$

for each $b \in \overline{B_{r\epsilon}}$, where $V_i - b$ is the translate of V_i by b and $V_i - b \cap A_{\delta,r}$ denotes the restriction of $V_i - b$ to $A_{\delta,r}$. Choose $b_i \in \overline{B_{r\epsilon}}$ such that

$$\delta_i(b_i) = \min_{b \in \overline{B_{r\epsilon}}} \delta_i(b). \tag{105}$$

Put $\delta_i = \delta_i(b_i)$.

Lemma 22. $\delta_i \to 0$ *as* $i \to \infty$.

Proof. Since $\delta_i \leq \delta_i(0)$ for all integers i it will suffice to prove $\delta_i(0) \to 0$ as $i \to \infty$. Let $\delta > 0$. By (99) we have $E(V \setminus \overline{B_\delta}) < \epsilon/2$. Since $V_i \to V$ as $i \to \infty$ there exists i_δ such that if $i > i_\delta$ then $E(V_i \setminus \overline{B_\delta}) < \epsilon/2$. If $E(V_i \setminus \overline{B_\delta}) < \epsilon/2$ then $\delta_i(0) < \delta$. Hence $\delta_i(0) \to 0$ as $i \to \infty$. This completes the proof of Lemma 22. $\square$

Lemma 23. *There exists a subsequence* $\{\delta_{i_j\#}(V_{i_j} - b_{i_j})\}_{j=1}^{\infty}$ *converging to some* $\Omega_{\mathbb{R}^6}$*-special Lagrangian varifold* W *with* $\partial\overrightarrow{W} = 0$ *in* $(\mathbb{R}^6, g_{\mathbb{R}^6})$.

Proof. Let $R > 1$ be arbitrary. By Theorem 20 we have

$$\limsup_{i=1,2,\ldots} d(\delta_{i\#}^{-1}(V_i - b_i), C \cap A_{R,2R}) < \infty. \tag{106}$$

Choose a compactly supported smooth function $\chi : (0, \infty) \to [0, 1]$ such that

$$\|\delta_{i\#}^{-1}(V_i - b_i)\|(B_R) \leq \overrightarrow{\delta_{i\#}^{-1}(V_i - b_i)}(\chi \operatorname{Re} \Omega), \tag{107}$$

where $\overrightarrow{V}$ is the current associated with a special Lagrangian varifold V; see Section 1. We shall now use the technique in [9, Section 3]. Put $\Omega_i = \delta_{i\#}^{-1}\Omega$. By [9, Proposition 3.3] and the proof of [9, Proposition 3.4] there exists a 2-form Ψ_i on $\mathbb{R}^6 \setminus \{0\}$ such that

$$\operatorname{Re} \Omega_i = d(3^{-1}|x|^3\Psi_i), \tag{108}$$

$$|\Psi_i - \Psi_{S^5}|_{\mathrm{cyl}} \leq |\Omega_i - \Omega_{\mathbb{R}^6}|, \tag{109}$$

where $\Psi_{S^5} = ((x^1\partial_1 + \cdots + x^6\partial_6) \lrcorner \Omega_{\mathbb{R}^6})|_{S^5}$. By (108) we have

$$\|\delta_{i\#}^{-1}(V_i - b_i)\|(B_R) \leq \overrightarrow{\delta_{i\#}^{-1}(V_i - b_i)}(-d\chi \wedge 3^{-1}|x|^3\Psi) \tag{110}$$

$$= \int_{\operatorname{spt}\|\delta_{i\#}^{-1}(V_i - b_i)\| \cap A_{R,2R}} -d\chi \wedge 3^{-1}|x|^3\Psi. \tag{111}$$

By (106), (107) and (110)–(111) there exists $k_R > 0$ independent of i such that

$$\|\delta_{i\#}^{-1}(V_i - b_i)\|(B_R) \leq k_R \sup_{r \in (R,2R)} \int_{C \cap S^5} \Psi|_{\partial B_r}. \tag{112}$$

By (109) and (112) there exist constants $k'_R, k''_R, k'''_R > 0$ independent of i such that

$$\|\delta_{i\#}^{-1}(V_i - b_i)\|(B_R) \leq k'_R + k''_R \int_{C \cap S^5} \Psi_{S^5} = k'''_R < \infty. \tag{113}$$

Hence there exists a subsequence $\{\delta_{i_j\#}(V_{i_j}-b_{i_j})\}_{j=1}^{\infty}$ converging to some varifold W of dimension 3 in $\mathbb{R}^6$.

It remains to prove that W is an Ω-special Lagrangian varifold with $\partial\overrightarrow{W}=0$ in $(\mathbb{R}^6, g_{\mathbb{R}^6})$. Since $\Omega_i \to \Omega$ as $i\to\infty$ in the local C^0 topology, W is an $\Omega_{\mathbb{R}^6}$-special Lagrangian varifold in $\mathbb{R}^6$. On the other hand since $\partial(\overrightarrow{\delta_{i\#}^{-1}(V_i-b_i)})=0$ we have $\partial\overrightarrow{W}=0$. This completes the proof of Lemma 23. □

Let W be as in Lemma 23. By (104) we have $E(\delta_{i_j\#}V_{i_j}-b_{i_j}\setminus\overline{B_1})=\epsilon/2$ for any integer $j\geq 1$. Letting $j\to\infty$ we obtain

$$E(W\setminus\overline{B_1})=\epsilon/2. \tag{114}$$

One can therefore use the technique in the previous section. In a similar way to the proof of Theorem 9 one finds $\gamma\in SU_3$ such that W is tangent to $\gamma(C)$ with multiplicity 1 at infinity in $\mathbb{C}^3$.

We shall now prove $E(W-a)\geq\epsilon/100$ for every $a\in\mathbb{R}^6$, which will complete the proof of Theorem 19. Let $a\in\mathbb{R}^6$. Then $\delta_{i\#}^{-1}V_i-a$ restricted to $A_{1,\delta_i^{-1}r}$ is represented with multiplicity 1 by a submanifold M_i of $A_{1,\delta_i^{-1}r}$. By (114) and the definition of $\delta_i=\delta_i(b_i)$ we have

$$E(M_i\cap A_{1,\delta_i^{-1}r})\geq\epsilon/2 \tag{115}$$

whenever i is sufficiently large. We shall use Ψ_i satisfying (108) and (109). By [9, Proposition 3.4] there exists $k>0$ independent of i such that

$$\int_{M_i\cap\partial B_{\delta_i^{-1}r}}\Psi_i-\int_{M_i\cap\partial B_1}\Psi_i\geq E(M_i)-k\int_{M_i}|\Omega_i-\Omega_{\mathbb{R}^6}|d\,\mathrm{vol}_{\mathrm{cyl}}, \tag{116}$$

where $\mathrm{area}_{\mathrm{cyl}}(M_i)$ is the area of M_i with respect to $g_{\mathrm{cyl}}=g_{\mathbb{R}^6}/|x|^2$. In what follows $k',k'',\ldots$ will denote constants independent of i. By Theorem 20 we have

$$\int_{M_i}|\Omega_i-\Omega_{\mathbb{R}^6}|d\,\mathrm{vol}_{\mathrm{cyl}}\leq k'\int_1^{\delta_i^{-1}r}|\Omega_i|_{\partial B_\rho}-\Omega_{\mathbb{R}^6}|_{\partial B_\rho}|d\rho/\rho. \tag{117}$$

Since $|\Omega_i|_{\partial B_\rho}-\Omega_{\mathbb{R}^6}|_{\partial B_\rho}|\leq k''\delta_i\rho$ we have

$$\int_{M_i}|\Omega_i-\Omega_{\mathbb{R}^6}|d\,\mathrm{vol}_{\mathrm{cyl}}\leq\int_1^{\delta_i^{-1}r}k''\delta_i\rho d\rho/\rho \tag{118}$$

$$\leq(\delta_i^{-1}r-1)k''\delta_i \tag{119}$$

$$\leq k'''r. \tag{120}$$

By (115)–(120) we have

$$\int_{M_i\cap\partial B_{\delta_i^{-1}r}}\Psi_i-\int_{M_i\cap\partial B_1}\Psi_i\geq\epsilon/4-k''''r\geq\epsilon/8 \tag{121}$$

whenever $r>0$ is sufficiently small. By Theorem 20 we have

$$\int_{M_i\cap\partial B_{\delta_i^{-1}r}}\Psi_i=\int_{M_i\cap\partial B_{\delta_i^{-1}r}}\Psi_i-\Psi_{S^5}+\Psi_{S^5} \tag{122}$$

$$\leq k'''''r+k''''''\eta^{k'''''''}+\int_{C\cap S^5}\Psi_{S^5} \tag{123}$$

We have

$$\int_{M_i\cap\partial B_1}\Psi_i=\int_{M_i\cap\partial B_1}\Psi_i-\Psi_{S^5}+\Psi_{S^5}\leq k''''''''\delta_i+\int_{M_i\cap\partial B_1}\Psi_{S^5} \tag{124}$$

By (121)–(124) we have

$$\int_{C\cap S^5}\Psi_{S^5}-\int_{M_i\cap\partial B_1}\Psi_{S^5}\geq\epsilon/16 \tag{125}$$

whenever $r,\eta,\delta_i>0$ are sufficiently small. Letting $i\to\infty$ we obtain

$$\int_{C\cap S^5}\Psi_{S^5}-\int_{\mathrm{spt}\,\|W-a\|\cap\partial B_1}\Psi_{S^5}\geq\epsilon/16. \tag{126}$$

By (126) and [9, Proposition 3.4] we have

$$E(W-a\setminus\overline{B_1})\geq\epsilon/100. \tag{127}$$

This completes the proof of Theorem 19. □

4. Proof of Theorem 8

We first recall the statement of Theorem 8:

Let W be an $\Omega_{\mathbb{R}^6}$-special Lagrangian varifold with $\partial\overrightarrow{W}=0$ in $\mathbb{R}^6$. Suppose that W is tangent to C with multiplicity 1 at infinity in $\mathbb{R}^6$. Then W is represented with multiplicity 1 by C or there exists $a>0$ such that W is represented with multiplicity 1 by L up to the action of $SU_3\ltimes\mathbb{R}^6$. Here L is as in (11) and it depends on $a>0$.

This section will be devoted to a proof of Theorem 8, the statement above. We begin with:

Definition 24. We say that V is strongly tangent to C with multiplicity 1 at infinity in $\mathbb{C}^3$ if there exists $R>0$ and a normal vector field v on $C\setminus\overline{B_R}$ such that $V\setminus\overline{B_R}$ is represented with multiplicity 1 by the graph of v with $v=O(|x|^{-1})$ where $O(\bullet)$ is as in Definition 26 below.

Put

$$(W-a)(f)=\int_{\mathbb{R}^6\times G_3(\mathbb{R}^6)}f(x-a,S)dW(x,S) \tag{128}$$

whenever f is a continuous function with compact support in $\mathbb{R}^6$.

We first prove:

Lemma 25. *Let W be an $\Omega_{\mathbb{R}^6}$ special Lagrangian varifold with $\partial\overrightarrow{W}=0$ in $\mathbb{R}^6$. Suppose W is tangent to C with multiplicity 1 at infinity in $\mathbb{C}^3$. Then there exists $a\in\mathbb{C}^3$ such that $W-a$ is strongly tangent to C with multiplicity 1 at infinity in the sense of Definition 24.*

Let C be as in (4). Let $x:C\setminus\{0\}\to\mathbb{C}^3$ be the inclusion map. Let $x^*g_{\mathbb{R}^6}$ be the induced metric on $C\setminus\{0\}$. Let

$$\Delta_C:C^\infty(C\setminus\{0\})\to C^\infty(C\setminus\{0\}) \tag{129}$$

be the Laplacian of $(C\setminus\{0\},x^*g_{\mathbb{R}^6})$. Put $|z|=\sqrt{|z_1|^2+|z_2|^2+|z_3|^2}$ for each $(z_1,z_2,z_3)\in\mathbb{C}^3$. Put $t=\log|x|:C\setminus\{0\}\to\mathbb{R}$. Put $S^5=\{z\in\mathbb{R}^6:|z|=1\}$. The map $x\to(t,x/|x|)$ is a diffeomorphism $C\setminus\{0\}\cong\mathbb{R}\times(C\cap S^5)$. We have

$$-|x|^2\Delta_C=\partial_t^2+\partial_t-\Delta_{C\cap S^5}, \tag{130}$$

where $\partial_t=\partial/\partial t$ and

$$\Delta_{C\cap S^5}:C^\infty(C\cap S^5)\to C^\infty(C\cap S^5); \tag{131}$$

this is the Laplacian with respect to $g_{\mathbb{R}^6}|_{C\cap S^5}$. Let Λ be the set of all $\lambda\in\mathbb{R}$ such that $\lambda(\lambda+1)$ is an eigenvalue of (131). As in [11, p11] we have

$$\Lambda\cap[-1,2]=\{-1,0,1,2\}. \tag{132}$$

Put

$$E_\lambda = \{f \in C^\infty(C \cap S^5) : \Delta_{C\cap S^5} f = \lambda(\lambda+1)f\} \tag{133}$$

for each $\lambda \in \Lambda$. As in [11, p11] we have

$$E_1 = \{(a_1x^1 + \cdots + a_6x^6)|_{C\cap S^5} : a_1, \ldots, a_6 \in \mathbb{R}\}. \tag{134}$$

Let $R > 0$. Let $u \in C^\infty(C \setminus \overline{B_R})$.

Definition 26. Let $p \in \mathbb{R}$. We write $u(x) = O(|x|^p)$ if

$$\limsup_{|x|\to\infty} |\nabla^k u|/|x|^{p-k} < \infty \tag{135}$$

for every integer $k \geq 0$, where ∇ is the covariant derivarive on $C \setminus \overline{B_R}$ induced from $g_{\mathbb{R}^6}$ and $|\bullet|$ is the norm induced from $g_{\mathbb{R}^6}$.

Lemma 27. *Let $p, q \in \mathbb{R}$ with $-1 < q < p$. Suppose*

$$u = O(|x|^p), \tag{136}$$

$$\Delta_C u = O(|x|^{q-2}). \tag{137}$$

Then for all $\lambda \in \Lambda \cap (q, p]$ there exist $f_\lambda \in E_\lambda$ such that

$$u = \sum_{\lambda \in \Lambda\cap(q,p]} e^{\lambda t} f_\lambda + O(|x|^q). \tag{138}$$

Proof. Let $0 = \gamma_0 \leq \gamma_1 \leq \ldots$ be the eigenvalues of $\Delta_{C\cap S^5}$. Let $\{v_0, v_1, \ldots\}$ be a basis of $L^2(C \cap S^5)$ such that $\Delta_{C\cap S^5} v_i = \gamma_i v_i$ for each integer $i \geq 0$. Put $\alpha_i = (-1 + \sqrt{1+4\gamma_i})/2, \beta_i = (-1 - \sqrt{1+4\gamma_i})/2$ for each integer $i \geq 0$. Let $T > \log R$. Put

$$u_i' = e^{\beta_i(t-T)}(u, v_i)|_{t=T} - e^{\beta_i t}\int_T^t e^{(\alpha_i-\beta_i)r}\int_r^\infty e^{-\alpha_i s}(\Delta_C u, v_i)dsdr \tag{139}$$

for each i with $\alpha_i > q$, where $(\bullet, \bullet)$ is the $L^2(C \cap S^5)$ inner product. Put

$$u_i' = e^{\beta_i(t-T)}(u, v_i)|_{t=T} + e^{\beta_i t}\int_T^t e^{(\alpha_i-\beta_i)(r-T)}(\partial_t e^{-\beta_i t}(u, v_i))|_{t=T}dr \tag{140}$$

$$+ e^{\beta_i t}\int_T^t e^{(\alpha_i-\beta_i)r}\int_T^t e^{-\alpha_i s}(\Delta_C u, v_i)dsdr \tag{141}$$

for each i with $\alpha_i \leq q$, where $(\bullet, \bullet)$ is the $L^2(C \cap S^5)$ inner product. Putting $u' = \sum_{i=0}^\infty u_i' v_i$ we have $u' \in C^\infty(C \setminus \overline{B_R})$, $\Delta_C u' = \Delta_C u$ and $u' = O(|x|^q)$. By (136) we have $u - u' = O(|x|^p)$. Hence for all $\lambda \in \Lambda$ with $\lambda \leq p$ there exist $f_\lambda \in E_\lambda$ such that $u - u' = \sum_{\lambda\in\Lambda,\lambda\leq p} e^{\lambda t} f_\lambda$. This proves (138). $\square$

Let $w \in C^\infty(C \setminus \overline{B_R})$. Consider the map $x + w : C \setminus \overline{B_R} \to \mathbb{C}^3$. Let $(x + w)^* g_{\mathbb{R}^6}$ be the induced metric on $C \setminus \overline{B_R}$. Let

$$\Delta_w : C^\infty(C \setminus \overline{B_R}) \to C^\infty(C \setminus \overline{B_R}) \tag{142}$$

be the Laplacian with respect to $(x + w)^* g_{\mathbb{R}^6}$. By calculation we have:

Lemma 28. *Suppose $u \in C^\infty(C \setminus \overline{B_R})$, $|u| = O(|x|^p)$ and $|w| = O(|x|^q)$. Then*

$$\Delta_w u = \Delta_C u + O(|x|^{2(p-1)+q-2}). \tag{143}$$

Using Lemmata 27 and 28 we give:

Proof of Lemma 25. There exists a normal vector field w on $C \setminus \overline{B_R}$ with respect to $g_{\mathbb{R}^6}$ such that $\operatorname{spt}\|W\| \setminus \overline{B_R} = \operatorname{graph} w$ with $\|w\| = O(|x|^p)$ for some $p < 1$. By Definition 24 it will suffice to find a constant a such that

$$w - a = O(|x|^{-1}). \tag{144}$$

We shall regard w as a $\mathbb{C}^3$-valued function on $C \setminus \overline{B_R}$. Since $\operatorname{graph} w$ is a minimal surface, we have $\Delta_w w = 0$. By Lemma 28 we have

$$w = O(|x|^p), p < 1 \Longrightarrow \Delta_C w = O(|x|^{3(p-1)-1}). \tag{145}$$

By (145), Lemma 27 and (132) we have $w = O(|x|^{3^n(p-1)+1})$ for every integer $n \geq 0$ with $3^n(p-1) \geq -1$. Since $p < 1$ there exists N such that $3^{N+1}(p-1) < -1$. Put $q = 3^{N+1}(p-1) + 1 < 0$. By (145), Lemma 27 and (132) there exists a constant a such that $w = a + O(|x|^q)$.

By Lemma 28 we have

$$w - a = O(|x|^q), q < 1 \Longrightarrow \Delta_C(w - a) = O(|x|^{3(q-1)-1}). \tag{146}$$

By (146), Lemma 27 and (132) we have $w - a = O(|x|^{3^n(q-1)+1})$ for every integer $n \geq 0$ with $3^n(q-1) \geq -1$. Since $q < 0$ there exists N such that $3^{N+1}(q-1) < -1$. Put $r = 3^{N+1}(p-1)+1 < -1$. By (146), Lemma 27 and (132) there exists $f_{-1} \in E_{-1}$ such that $w - a = f_1 + O(|x|^r)$. This proves (144). We have therefore completed the proof of Lemma 25. □

Define $\mu : \mathbb{C}^3 \to \mathbb{R}^2$ by

$$\mu(z_1, z_2, z_3) = (|z_1|^2 - |z_2|^2, |z_1|^2 - |z_3|^2). \tag{147}$$

Define $f : \mathbb{C}^3 \to \mathbb{R}^3$ by

$$\begin{aligned} f(z_1, z_2, z_3) &= (\mu(z_1, z_2, z_3), \operatorname{Im} z_1 z_2 z_3) && (148) \\ &= (|z_1|^2 - |z_2|^2, |z_1|^2 - |z_3|^2, \operatorname{Im} z_1 z_2 z_3). && (149) \end{aligned}$$

Put $E_1 = \{(a, a, 0) \in \mathbb{R}^3 : a \geq 0\}$, $E_2 = \{(-a, 0, 0) \in \mathbb{R}^3 : a \geq 0\}$, $E_3 = \{(0, -a, 0) \in \mathbb{R}^3 : a \geq 0\}$ and

$$Y = E_1 \cup E_2 \cup E_3. \tag{150}$$

Put $0 = (0, 0, 0) \in Y$.

Let $y \in Y$. Put

$$L_y = f^{-1}(y) \cap \{(z_1, z_2, z_3) \in \mathbb{C}^3 : \operatorname{Re} z_1 z_2 z_3 \geq 0\}. \tag{151}$$

We have $f^{-1}(y) = L_y \cup -L_y$. Notice that L_y is connected. If $y = 0$ then L_0 is a cone in $\mathbb{C}^3$ with $L_0 \cap S^5$ diffeomorphic to T^2. If $y \in Y \setminus \{0\}$ then L_y is a submanifold of $\mathbb{C}^3$ diffeomorphic to $S^1 \times \mathbb{R}^2$.

If $y' \in \mathbb{R}^3 \setminus Y$ then $f^{-1}(y')$ is a connected closed submanifold of $\mathbb{C}^3$ diffeomorphic to $\mathbb{R} \times T^2$.

By [8, III.3.A, Theorem 3.1] $f^{-1}(y')$, L_y and $L_0 \setminus \{0\}$ are special Lagrangian submanifolds of $\mathbb{C}^3$.

By Lemma 25 it will suffice to prove the following theorem in order to prove Theorem 8:

Theorem 29. *Let V be an $\Omega_{\mathbb{R}^6}$-special Lagrangian varifold with $\partial\overrightarrow{V} = 0$ in $\mathbb{R}^6$. Suppose that V is strongly tangent to C with multiplicity 1 at infinity in $\mathbb{C}^3$ in the sense of Definition 24. Then there exists $y \in Y$ such that V is represented with multiplicity 1 by L_y.*

Let μ be as in (147).

Lemma 30. $\operatorname{grad}_{T_z V} \mu(z) = 0$ *for $\|V\|$-almost every $z \in \mathbb{C}^3$.*

Proof. μ is a moment map of the T^2-action

$$(t_1, t_2) : (z_1, z_2, z_3) \to (t_1 z_1, t_2 z_2, t_1^{-1} t_2^{-1} z_3) \tag{152}$$

on $\mathbb{C}^3$, where $T^2 = \{(t_1, t_2) \in \mathbb{C}^2 : |t_1| = |t_2| = 1\}$. By [11, Lemma 3.4] we have

$$\operatorname{div}_{TV} \operatorname{grad}_{TV} \mu = 0. \tag{153}$$

Since V is strongly tangent to C with multiplicity 1 at infinity in $\mathbb{C}^3$ in the sense of Definition 24 we have

$$\mu|_{\operatorname{spt}\|V\| \setminus \overline{B_R}} = O(|x|^0). \tag{154}$$

Lemma 31. *Let $v \in C^\infty(C \setminus \overline{B_R})$ with*

$$v = O(|x|^{-1}). \tag{155}$$

Suppose $u \in C^\infty(\operatorname{graph} v)$, $u = O(|x|^0)$ and $\Delta_v u = 0$, where Δ_v is as in (142). *Then there exists a constant c such that*

$$u - c = O(|x|^{-1}). \tag{156}$$

Proof. By (155) and Lemma 28 we have

$$\Delta_C u = O(|x|^{-6}). \tag{157}$$

By (157), Lemma 27 and (132) there exists a constant c such that $u = c + O(|x|^{-1})$. This proves Lemma 31. □

By Lemma 31 there exists a constant c such that

$$\mu - c|_{\operatorname{spt}\|V\| \setminus \overline{B_R}} = O(|x|^{-1}). \tag{158}$$

Since V has first variation 0 we have

$$\int_{\mathbb{C}^3} \operatorname{div}_{TV}(\chi(\mu - c) \operatorname{grad}_{TV} \mu) d\|V\| = 0 \tag{159}$$

whenever χ is a smooth function with compact support in $\mathbb{C}^3$. By (158) we have

$$\lim_{\chi \to 1} \int_{\mathbb{C}^3} (\mu - c) \operatorname{tr}(d\chi|_{TV} \otimes \operatorname{grad}_{TV} \mu) d\|V\| = 0, \tag{160}$$

where 1 is the constant function on $\mathbb{C}^3$. By (159), (153) and (160) we have

$$\int_{\mathbb{C}^3} |\operatorname{grad}_{TV} \mu|^2 d\|V\| = 0. \tag{161}$$

This proves Lemma 30. □

Let $f : \mathbb{C}^3 \to \mathbb{R}^3$ be as in (149). By Lemma 30 and the proof of [8, III.3.A, Theorem 3.1] we have:

Corollary 32. *$T_z V = \operatorname{Ker} df(z)$ for $\|V\|$-almost every $z \in \mathbb{C}^3$.*

From Corollary 32 we obtain:

Corollary 33. *There exist $R > 0$ and $y_V \in Y$ such that V restricted to $\mathbb{C}^3 \setminus \overline{B_R}$ is represented with multiplicity 1 by a submanifold contained in some fibre of f.*

Let Θ_V be the multiplicity of V. It is a $\mathcal{H}^3$-measurable function on $\mathbb{C}^3 = \mathbb{R}^6$ such that $\|V\| = \mathcal{H}^3 \llcorner \Theta_V$, where $\mathcal{H}^3$ is the Hausdorff measure of dimension 3 in $\mathbb{R}^6$.

Lemma 34. *Let L be a 3-dimensional submanifold of $\mathbb{C}^3 = \mathbb{R}^6$ with $T_z L = T_z V$ for $\|V\|$-almost every $z \in \mathbb{C}^3$. Then Θ_V is a locally constant function on L.*

Proof. Since V has first variation 0 we have

$$\int_L (\operatorname{div}_{TL} \xi) \Theta_V d\mathcal{H}^3 = 0 \tag{162}$$

whenever ξ is a vector field with compact support in L. Hence $d\Theta_V = 0$ as a distribution on L. It is therefore locally constant on L. □

This is a technique which was used in the proof of the constancy theorem; see [1, Theorem 4.6.(3)] or [17, Theorem 41.1].

Let $y' \in \mathbb{R}^3 \setminus Y$ and $y \in Y$. Let L_y be as in (151). The submanifolds $f^{-1}(y')$, L_y and $-L_y$ are connected. Hence by Corollaries 32 and 34 we have:

Corollary 35. *$\Theta_V|_{f^{-1}(y')}$, $\Theta_V|_{L_y}$ and $\Theta_V|_{-L_y}$ are constant functions.*

We shall use the fact that if $y \in Y$ then L_y is tangent to C with multiplicity 1 at infinity in $\mathbb{R}^6$ and if $y' \in \mathbb{R}^3 \setminus Y$ then $f^{-1}(y')$ is tangent to $C \cup -C$ at infinity in $\mathbb{R}^6$. Then from Corollaries 33 and 35 we obtain:

Corollary 36. *There exists $y_V \in Y$ such that $\Theta_V|_{L_{y_V}} = 1$ and $\Theta_V|_{L_y} = 0$ for every $y \in Y \setminus \{y_V\}$. Moreover $\Theta_V|_{-L_y} = 0$ for every $y \in Y$ and $\Theta_V|_{f^{-1}(y')} = 0$ for every $y' \in \mathbb{R}^3 \setminus Y$.*

By definition

$$\mathbb{C}^3 = \bigcup_{y'' \in \mathbb{R}^3} f^{-1}(y'') = \bigcup_{y \in Y} L_y \cup \bigcup_{y \in Y} -L_y \cup \bigcup_{y' \in \mathbb{R}^3 \setminus Y} f^{-1}(y'). \tag{163}$$

By Corollary 36 and (163) V is represented with multiplicity 1 by L_{y_V}. This completes the proof of Theorem 29. From Lemma 25 and Theorem 29 we obtain Theorem 8.

5. Completion of the Proof of Theorem 6

In this section we complete the proof of Theorem 6.

Let L be as in (11), which depends on $a > 0$. Choose $a > 0$ such that $E(L \setminus \overline{B_1}) = \epsilon_{19}/100$, where $\epsilon_{19} > 0$ is as in Theorem 19. Using Theorems 8, 19, 20 and (14) one can prove:

Lemma 37. *Let $B \supset B'$ be sufficiently small neighbourhoods of x where x is the singular point of M in X. Put $B'^{\epsilon} = \{x \in X : \inf_{y \in B'} \operatorname{dist}(x, y) < \epsilon\}$ for each $\epsilon > 0$. Let B_R be the ball of radius $R > 0$ centred at 0 in $\mathbb{R}^6$. Then for any $\epsilon, R > 0$ there exists a neighbourhood $\mathcal{U}$ of M in $\mathcal{V}$ such that $\mathcal{U} \in \mathcal{M} \cup \mathcal{N}$ and such that if $N \in \mathcal{U} \cap \mathcal{N}$ then the following holds: there exist normal vector fields v, v' on $L \cap B_R, M \setminus \overline{B'}$ on $(\mathbb{R}^6, g_{\mathbb{R}^6}), (X, g)$ respectively such that $\operatorname{graph} v' \supset N \setminus \overline{B'^{\epsilon}}$ and $\operatorname{graph} v \supset \lambda\alpha(N \cap B) \cap B_{R-\epsilon}$ for some $\lambda > 0$ and $\alpha \in SU_3 \ltimes \mathbb{R}^6$ with $\operatorname{dist}(1, \alpha) < \epsilon$, where $N \cap B$ is regarded as a subset of $\mathbb{R}^6$ so that we may define $\lambda\alpha(N \cap B)$, and such that $\|v\|_{C^2} < \epsilon, \|v'\|_{C^2} < \epsilon$, where $\|v\|_{C^2}$ is the norm induced from $g_{\mathbb{R}^6}$ and $\|v'\|_{C^2}$ is the norm induced from (X, g). Here g is as in (8).*

We have here used a normal coordinate of (X, g) at the singular point of M but we shall use a Darboux coordinate with respect to the Kähler form of X in what follows.

Let $\mathcal{U}$ be as in Lemma 37. Define $F : \mathcal{U} \to \mathbb{R} \times \mathcal{M}$ as follows:

Let $P \in \mathcal{U}$. Let B be as in Lemma 37. By Weinstein's theorem we may regard $P \setminus \overline{B}$ as the graph of a closed 1-form u on $M \setminus \overline{B}$. We have

$$[u] \in H^1(M \setminus \overline{B}; \mathbb{R}) \cong H^1(M \setminus \{x\}; \mathbb{R}), \tag{164}$$

where x is the singular point of M. Let f_M be as in (13). By [13, Lemma 10.1] $\dim \operatorname{Im}(f_M : H^1(M \setminus \{x\}; \mathbb{R}) \to H^1(C \cap S^5; \mathbb{R})) = 1$. Hence there exists a unique

$s \in \mathbb{R}$ such that $f_M[u] = sa(L)$, where $a(L)$ is as in (12). Put $F_1(P) = s$. By [10, Lemma 4.5] we have $F_1(P) = 0$ whenever $P \in \mathcal{M}$. By Lemma 37 we have $F_1(P) > 0$ whenever $P \in \mathcal{N}$. By [11, Corollary 6.11] there exists a vector space isomorphism $\operatorname{Ker} f_M \cong T_M\mathcal{M}$. Choose $b \in H^1(M \setminus \{x\}; \mathbb{R})$ such that $f_M(b) = a(L)$. We may assume that $\mathcal{U}$ is so small that $[u] - \lambda b \in \operatorname{Ker} f_M$ defines some $Q \in \mathcal{M}$. Put

$$F_2(P) = Q \in \mathcal{M}, \tag{165}$$

$$F(P) = (F_1(P), F_2(P)) \in \mathbb{R} \times \mathcal{M}. \tag{166}$$

This is the definition of $F : \mathcal{U} \to \mathbb{R} \times \mathcal{M}$.

By definition

$$F(\widehat{M}) = (0, \widehat{M}) \tag{167}$$

for every $\widehat{M} \in \mathcal{U} \cap \mathcal{M}$.

One must choose $b \in H^1(M \setminus \{x\}; \mathbb{R})$ such that $f_M(b) = a(L)$ in order to define

$$G : (0, \epsilon) \times \mathcal{W} \to \mathcal{N} \tag{168}$$

as in (15). We use here the same b as in the definition of Q above. Then $F \circ G$ is an identity map.

Lemma 38. *$DF|_N : T_N\mathcal{N} \to T_{F(N)}(\mathbb{R}_{>0} \times \mathcal{M})$ is a vector space isomorphism for every $N \in \mathcal{U} \cap \mathcal{N}$.*

Proof. Since $F \circ G$ is an identity map, $DF|_N : T_N\mathcal{N} \to T_{F(N)}(\mathbb{R}_{>0} \times \mathcal{M})$ is onto. But by [13, Theorem 8.7] we have

$$\dim T_N\mathcal{N} = 1 + \dim T_{\widehat{M}}\mathcal{M}, \tag{169}$$

where $\widehat{M} = F_2(N)$. Thus $DF|_N : T_N\mathcal{N} \to T_{F(N)}(\mathbb{R}_{>0} \times \mathcal{M})$ is a vector space isomorphism. $\square$

Lemma 39. *$G \circ F|_{\mathcal{U}\cap\mathcal{N}}$ is an identity map.*

Proof. Let $N \in \mathcal{U}\cap\mathcal{N}$. There exists a closed 1-form u on N whose graph is $G\circ F(N)$. We shall first prove $[u] = 0 \in H^1(N; \mathbb{R})$. Let x be the singular point of M. Let U, U' be sufficiently small neighbourhoods of x in X. The inclusion maps $U \cap N \to N, N \setminus \overline{U'} \to N$ induces $H^1(N; \mathbb{R}) \to H^1(U \cap N; \mathbb{R}) \oplus H^1(N \setminus \overline{U'}; \mathbb{R})$, which is one-to-one. It will therefore suffice to prove $[u|_{N\cap U}] = [u|_{N\setminus\overline{U'}}] = 0$ in order to prove $[u] = 0$. We shall first prove $[u|_{N\setminus\overline{U'}}] = 0$. There exists a sufficiently small neighbourhood U'' of x in X and closed 1-forms v_1, v_2 on $M \setminus \overline{U''}$ with $[v_1] = [v_2] \in H^1(M \setminus \overline{U''}; \mathbb{R})$ such that $\operatorname{graph} v_1 \supset N \setminus \overline{U'}$ and $\operatorname{graph} v_2 \supset G\circ F(N) \setminus \overline{U'}$. One can find, using v_1 and v_2, a smooth function f on $N \setminus \overline{U'}$ such that $df = u|_{N\setminus\overline{U'}}$. Hence $[u|_{N\setminus\overline{U'}}] = 0$. We shall next prove $[u|_{N\cap U}] = 0$. Let B, λ, α be as in Lemma 37. There exists a 1-form w on L such that $\operatorname{graph} w = \lambda\alpha(N \cap B)$. It will suffice to prove that w is exact; it will then be easy to prove $[u|_{N\cap U}] = 0$ in a way similar to $[u_{N\setminus\overline{U'}}] = 0$. It will therefore suffice to prove $\int_\gamma w = 0$ for $\gamma \subset L$ whose homology class generates $H^1(L; \mathbb{Z}) \cong H^1(S^1 \times \mathbb{R}^2; \mathbb{Z}) \cong \mathbb{Z}$. Let $D \subset B$ be a disc of dimension 2 with $\partial D = \gamma$, and D' the image of D under the map $x \mapsto \exp_x w(x)$ on L. Then $\int_\gamma w = \int_{D'} \omega - \int_D \omega$ where ω is the symplectic form of $\mathbb{C}^3$. It will therefore suffice to prove $\int_{D'} \omega - \int_D \omega = 0$. Letting $a(\bullet)$ be as in (12) we have $\int_{D'} \omega - \int_D \omega = a(\lambda\alpha(N \cap B)) \cap c - a(L) \cap c$ for some $c \in H_1(C \cap S^5)$. But $a(\lambda\alpha(N \cap B)) = a(L)$ by the definition of λ. Hence $\int_{D'} \omega - \int_D \omega = 0$. This completes the proof of $[u|_{N\cap U}] = 0$. We have therefore proved $[u] = 0$.

Hence $u = df$ for some smooth function f on N. Notice that f satisfies an elliptic equation as in Joyce [12, Proposition 5.6] ($\theta = 0$ in his notation since N is special

Lagrangian). f is therefore constant by the maximum principle. Hence $df = 0$ and $N = G \circ F(N)$, which proves Lemma 39. $\square$

Corollary 40. *$F|_{\mathcal{U}\cap\mathcal{N}} : \mathcal{U} \cap \mathcal{N} \to \mathbb{R}_{>0} \times \mathcal{M}$ is a diffeomorphism onto its image with inverse $G|_{F(\mathcal{U}\cap\mathcal{N})}$.*

Proof. From Lemmata 38 and 39 we obtain Corollary 40. $\square$

Corollary 41. *$F : \mathcal{U} \to \mathbb{R}_{\geq 0} \times \mathcal{M}$ is a homeomorphism onto its image.*

Proof. $F : \mathcal{U} \to \mathbb{R}_{\geq 0} \times \mathcal{M}$ is continuous. By (167), Corollary 40 and Lemma 39 it is one-to-one. $\mathcal{U}$ is locally compact. $\mathbb{R}_{\geq 0} \times \mathcal{M}$ is Hausdorff. $F : \mathcal{U} \to \mathbb{R}_{\geq 0} \times \mathcal{M}$ is therefore a homeomorphism onto its image. $\square$

From (167), Corollary 40 and Corollary 41 we obtain Theorem 6.

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